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1989 J. Phys.: Condens. Matter 1 2771

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LETTER TO THE EDITOR

Concentration anisotropy and directionality in the dielectric breakdown problem on a square lattice

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Received 14 December 1988

Abstract. We propose a small-cell position-space renormalisation group approach to study the effects of concentration anisotropy and directionality in the dielectric breakdown problem on a two-dimensional square lattice. We show that, in spite of the anisotropies inherent in the problem, concentration anisotropy is irrelevant (in the renormalisation group sense) both for the undirected and directed cases, just as occurs in normal percolation. For respective bond concentrations x and y along the Cartesian directions, we remark on relevant features of the phase diagram of insulating and conducting phases in the xy plane. For $x = y = p$, and V_b the breakdown voltage, the phase diagram in the $p-V_b$ plane displays a line of fixed points on the $p = 1$ axis, whose physical meaning is discussed. Directional effects are included via 'directed insulators', an approximation which results in the appearance of a mixed (non-conducting) phase consisting of conductors and 'mismatched' directed insulators.

The dielectric breakdown of a metal-loaded dielectric material, for example, is a macroscopic phenomenon which takes place through a series of microscopic failures (see e.g. Beale and Duxbury 1988). When a metal-loaded sample of linear dimension L is subjected to an electric field, the potential difference across the material will concentrate in the spatial regions filled with dielectric. Thus, for a given externally applied potential difference V , the local fields within the dielectric may be substantially higher than if the material were not metal-loaded. If the dielectric can only withstand a certain maximum local field before failing and becoming a conductor, raising the external potential difference to a breakdown value V_b will induce failure, initially at the points where the local field is highest, and propagating until a macroscopic breakdown occurs by the formation of a conducting path through the material.

Assuming the metallic impurities to be randomly distributed, standard percolation theory (Essam 1980, Stauffer 1985) tells us that the mixture will be conducting for impurity concentrations $c > c_c$, the percolation threshold. As $c \rightarrow c_c$ from below, the breakdown electric field $E_c (\equiv V_b/L)$ vanishes as $(c_c - c)^{t'}$. It has been proposed (Duxbury *et al* 1986, Beale and Duxbury 1988) that $t' = \nu$, the correlation-length exponent of the percolation problem. Qualitative differences arise depending on whether the problem is defined on a lattice or in the continuum (Lobb *et al* 1987 and references therein, Chakrabarti *et al* 1988); here, we shall concentrate on lattice models, in which the bonds can be occupied either by an insulator (dielectric) and probability $p \equiv 1 - c$, or by a conductor (metal) with probability c . Note that macroscopic insulation requires

that *both* (a) insulators percolate and (b) conductors do not percolate; also, at the critical point of interest $c = c_c$ the statistics of insulator clusters will not, in general, be the same as at the insulator percolation point $p = p_c$ (which is not relevant for the present problem). The square lattice, with which we shall deal here, is then an exception because in this case the bond percolation threshold equals $\frac{1}{2}$ exactly (Essam 1980).

For a given concentration of conductors below c_c , the breakdown voltage will depend strongly on the geometrical distribution of metallic elements: the 'most critical defect' (i.e. the one for which the breakdown voltage is the smallest) (Duxbury *et al* 1986, Beale and Duxbury 1988) will have the shape of a line parallel to the applied field. This is true for any space dimensionality d ; in the similar problem of random fuse networks (which, in $d = 2$ is analogous to the dielectric breakdown problem discussed here (Beale and Duxbury 1988)) the most critical defect is a $(d - 1)$ dimensional hyperplane of absent bonds (insulators, in that case) perpendicular to the applied field (Duxbury *et al* 1986, 1987). On the other hand, a line of metallic defects perpendicular to the applied field will have a negligible effect in lowering the breakdown voltage (relative to that for the pure dielectric). This inherently anisotropic feature of the problem leads one to ask how the characteristics of dielectric breakdown will change if additional anisotropies, such as concentration anisotropy or bond directionality are present. Apart from its interest as a problem in lattice statistics in its own right, this competition of anisotropies may simulate relevant experimental situations in the similar problem of mechanical fracture (see e.g. Sahimi and Goddard 1986 and references therein for typical problems in this field). For example, while the external stress in the mechanical problem is analogous the voltage in dielectrics, the gravitational field may provide a directional bias, and concentration anisotropies may arise in connection with anisotropic shapes of grain boundaries.

In what follows, we develop a small-cell position-space renormalisation group (PSRG) to deal with concentration anisotropy and directional effects in the dielectric breakdown problem on a two-dimensional square lattice. We make use of the physical ideas which form the 'minimum gap approximation' (Stinchcombe *et al* 1986), in which it is assumed that the breakdown voltage in a configuration can be approximated by the minimum number of insulating bonds (gap) in all the paths that cross the configuration from top to bottom. Chakrabarti *et al* (1987) treated the isotropic, undirected problem within this approximation both in $d = 2$ and 3, also in a small-cell PSRG scheme, with fairly good results as regards critical exponents and concentrations. They used in $d = 2$ the H-shaped cell of figure 1(a) (and its counterpart in $d = 3$) which has proved reliable in PSRG studies of standard percolation (see e.g. Reynolds *et al* 1980 and references therein). However, this type of cell is not suitable for taking into account anisotropy effects, at least in standard percolation, as shown by Oliveira (1982) for the anisotropic undirected problems and by Chame *et al* (1986) for the directed case. These authors show that the H-shaped cell gives excessive weight to one-dimensional paths, and this causes the crossover between one- and two-dimensional behaviour to be wrongly predicted as taking place at the isotropic point (instead of at the extreme anisotropic limit, as is known to be true for standard percolation). To remedy this, they propose 'tilted' cells (depicted in figures 1(b) and 1(c)), in which it is argued that one- and two-dimensional paths are treated more equitably. Indeed, with this scheme the correct crossover behaviour is obtained.

Here, we shall use the cells proposed by Oliveira (1982) and Chame *et al* (1986), with suitable adaptations for the dielectric breakdown problem (see below). The underlying assumption is that, as in normal percolation, no undue privilege is given either to

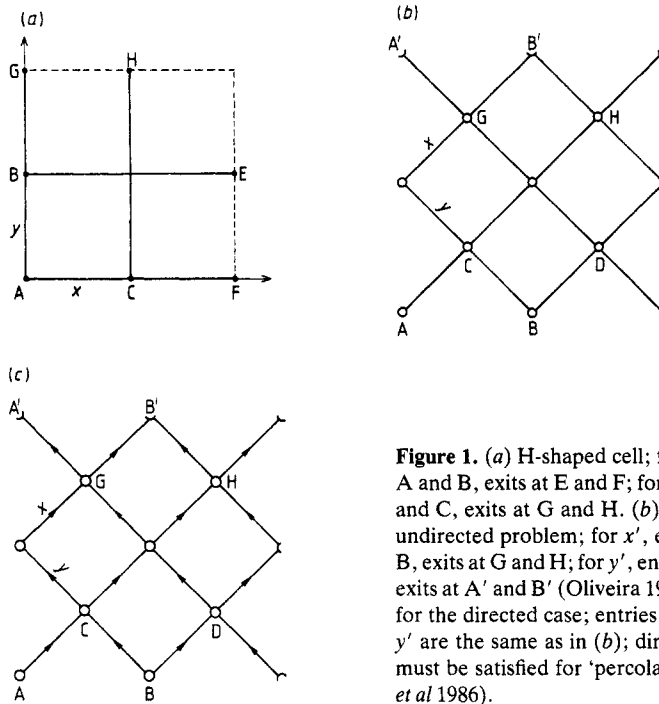


Figure 1. (a) H-shaped cell; for x' , entries are at A and B, exits at E and F; for y' , entries are at A and C, exits at G and H. (b) 'Tilted' cell for the undirected problem; for x' , entries are at A and B, exits at G and H; for y' , entries are at C and D, exits at A' and B' (Oliveira 1982). (c) 'Tilted' cell for the directed case; entries and exits for x' and y' are the same as in (b); directional constraints must be satisfied for 'percolating' paths (Chame *et al* 1986).

one- or to two-dimensional paths; thus the crossover behaviour emerging from our calculations will be related to the actual physical properties of the problem in study, not to an *a priori* bias.

The specific features of dielectric breakdown are incorporated as in Chakrabarti *et al* (1987): for given probabilities of occupation of horizontal (x) or vertical (y) bonds by insulators, we count for the renormalised x' and y' all configurations which span the cell (that is, cross it from 'entry' to 'exit' (see caption of figure 1) through only insulating bonds), and in which the conducting bonds (present with respective probabilities $(1 - x)$ and $(1 - y)$) do *not* span the cell. If, in the unrenormalised lattice, an insulator bond has breakdown voltage V_b , the breakdown voltage for a renormalised bond e.g. along the x direction is given through the quantity $x'V_b$, for which one sums over the contributions given by all configurations obeying the above rules. For each configuration with an intrinsic probability $f(x, y)$, its contribution is $(nV_b/b')f(x, y)$, where n is the number of insulating bonds in the minimum gap path of the configuration, and b' is the number of bonds in the shortest path crossing the cell ($b' = 2$ in the cell of figure 1(a), and 3 in those of figures 1(b) and 1(c)). In other words, it is actually the electric field (intensive quantity) which is being scaled, not the potential difference (extensive). This is already implicit in the treatment of Chakrabarti *et al* (1987). Since, by construction, $x' = R(x, y)$ and $y' = R(y, x)$ (see figure 1) the phase diagram obtained in the xy plane from the iteration of the recursion relations is bound to be symmetric about the $x = y$ line. The correlation-length exponent at the non-trivial fixed points is given by $\nu = \ln b / \ln \lambda_>$, where b is the scaling factor ($b = 2$ for the cell of figure 1(a), $b = 3$ for those of figures 1(b) and 1(c), see e.g. Chame *et al* 1988 and references therein), and $\lambda_>$ is the eigenvalue of the linearised recursion relations, related to the eigenvector pointing outward from the

critical surface. With $\lambda_V \equiv \partial V'_b / \partial V_b$ at the fixed point, the critical exponent t' is given by (Chakrabarti *et al* 1987)

$$t' = -\ln \lambda_V / \ln \lambda_\nu. \quad (1)$$

For the undirected anisotropic problem, we have the diagram depicted schematically in figure 2(a), about which we make the following remarks in order.

(i) The flow direction along the critical line ACB is from the extreme anisotropic points A and B, towards the isotropic point C; thus, concentration anisotropy is irrelevant (in the renormalisation-group sense) in dielectric breakdown.

(ii) The location of C is $x = y = 0.666$, compared to the expected $\frac{1}{2}$; this is a small-cell effect related to the restrictions imposed by the condition that conductors do not percolate.

(iii) At C, $\nu = 1.434$, $t' = 0.968$; these compare respectively with $\nu = \frac{4}{3}$ (den Nijs 1979) and $t' = 1.46 \pm 0.22$ (Beale and Duxbury 1988) and $t' = 1.31 \pm 0.05$ (Manna and Chakrabarti 1987). Although it is tempting to say that our results point towards $t' \neq \nu$, contrary to the reasoning of Beale and Duxbury (1988), we must refrain from attaching great significance to numerical values from small-cell PSRG calculations (even though our estimate for ν is quite good). Note also that, since $p_c = \frac{1}{2}$, we are justified in comparing our result for ν of the insulating clusters with values obtained for critical percolation clusters; as remarked above, it is only for this special case that the critical point of conducting clusters (which is the relevant feature) coincides with that of the insulators. We only have to recall that ν is the same on both sides of the transition (see e.g. Stanley 1971), which is implicit in scaling renormalisation-group theories.

(iv) At the one-dimensional critical points A and B, the crossover exponent $\varphi = 1$ exactly, as in standard percolation, and $t' = 1$, which again is an exact result (Stinchcombe *et al* 1986).

In figure 2(b) we have restricted ourselves to the isotropic problem ($x = y = p$) and have worked out the phase diagram in the $p-V_b/L$ plane. For $p < p_c$ (in our approximation, $p_c = 0.666$, see above) all lines flow toward the empty lattice point, in agreement with the fact that the system will be conducting, and insulating clusters do

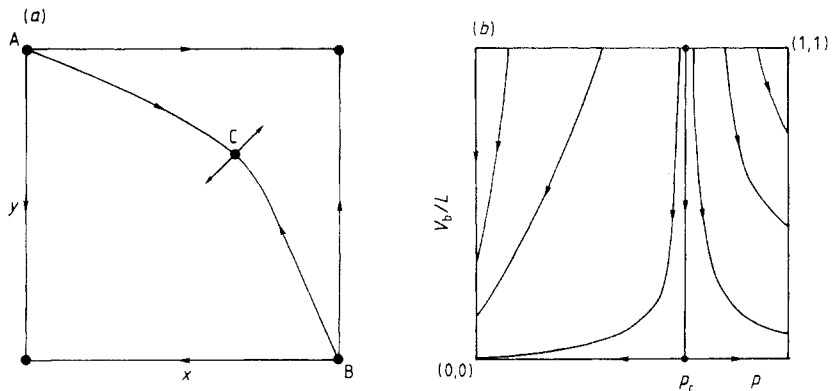


Figure 2. (a) Schematic phase diagram for insulating and conducting phases in the xy plane. For the undirected problem, C is at $x = y = 0.666$; for the directed case, C is at $x = y = 0.663$. (b) Schematic phase diagram and flow lines in the $p-V_b$ plane (see text). For the undirected problem, $p_c = 0.666$; for the directed case, $p_c = 0.663$.

not percolate; at $p = p_c$ the flow is toward $(p, V_b/L) = (p_c, 0)$, since at this point the system is both scale-invariant and unable to withstand any finite field without breaking down; for $p > p_c$ there is a line of fixed points at $p = 1$ which attracts all points as follows (see figure 2(b) for some schematic trajectories): for given V_b , the closer a point is initially to the pure limit $p = 1$, the closer still to $V_b/L = 1$ it will eventually become in reaching the line of fixed points. This illustrates that the breakdown voltage increases as the concentration of metallic defects $(1 - p)$ goes to zero, as is expected intuitively. Similar results have been obtained, in a (somewhat different) PSRG scheme, for the magnetisation versus temperature diagram of Potts ferromagnets (Chame *et al* 1988: their figure 2 is analogous to our figure 2(b), with the equivalence magnetisation $\equiv V_b/L$; temperature $\equiv 1 - p$). Note that there is no discontinuity in the flow lines as the point $(p, V_b/L) = (1, 1)$ is approached: this shows that our calculations do not reproduce the finite jump in V_b/L (from 1 to $\frac{1}{4}\pi$) when one single conducting impurity is added to an otherwise pure dielectric (Beale and Duxbury 1988). The reason for this is that the scaling assumptions underlying the renormalisation group (RG) are tantamount to assuming averaged translational invariance throughout the system; thus, the one-impurity problem is viewed by the RG as identical to that of exactly zero defects. It is also worth noting that, once again owing to scaling assumptions, the RG is not able to distinguish between the initial breakdown voltage V_1 (that for which the first bond collapses) and the complete breakdown voltage V_b (the one for which a conducting path throughout the material is formed). In this case, Beale and Duxbury (1988) have found that both quantities have essentially the same value, although there is a clear conceptual difference between them. In the random fuse network, however, the analogous breakdown voltages V_1 and V_b may differ for systems with a continuous distribution of breakdown strengths (Duxbury *et al* 1987).

Directionality effects can be included in a variety of ways; here, we have chosen what seemed to us the simplest one, namely via 'directed insulators'. By this we mean a bond which, when submitted to a potential difference (in a given direction) greater than the threshold V_b , will break down and become a diode (conducting current only in one direction). If the potential difference is along the opposite direction, the bond is assumed never to break down, no matter how strong the potential is. Thus on a square lattice we can have directed insulators along x bonds (with concentration x), along y bonds (with concentration y) or conductors (with overall concentration $1 - x - y$). The conductors are assumed to carry current both ways (undirected). For the cell of figure 1(c) we can then establish recursion relations for the renormalised concentrations x' , y' and for the renormalised breakdown voltage V'_b . The configurations counting for x' (or y') are those in which it is possible to cross the cell from entry to exit (see caption of figure 1) only through insulating bonds, now respecting directional constraints (see e.g. Tsallis and Redner 1983 for a discussion of this point), and in which the conducting bonds do *not* span. The recursion relation for V'_b is obtained through considerations similar to those for the undirected case (see above).

The phase diagram in the xy plane in this case is qualitatively similar to that obtained for the undirected problem (figure 2(a)). The following comments should be noted.

(i) We find that concentration anisotropy is irrelevant for dielectric breakdown with directed insulators; this is the same as for the undirected problem, and for normal percolation, both undirected (Oliveira 1982) and directed (Chame *et al* 1986).

(ii) The isotropic critical point C is at $x = y = 0.668$; compare with $p_c = 0.643 \pm 0.002$ (Blease 1977); Chame *et al* (1986) obtain $p_c = 0.5021$ for the same cell, in the problem

of normal directed percolation. Note that between $p = \frac{1}{2}$ (at which the concentration of isotropic conductors $1 - p$ becomes critical) and $p = 0.643$, one has an infinite cluster of 'directed insulators' with mismatched directionalities; since the insulators' breakdown voltage along the 'wrong' direction is assumed infinite, macroscopic conduction cannot take place, and the critical point is at the directed insulators' percolation threshold. Thus, there is a subtle difference compared with undirected dielectric breakdown.

(iii) At C, $\nu = 1.433$, $t' = 0.958$; compared with $\nu = 1.734 \pm 0.02$ (Kinzel 1983). The estimate of t' is very close to that for the undirected problem; in an experiment with resistors and diodes (which is, however, *not* equivalent to the situation discussed here), Benguigui (1988) has found $t' = 1.1 \pm 0.3$, also close to values reported for the undirected case.

(iv) As in the undirected case, the crossover exponent $\varphi = 1$ (see Chame *et al* 1986) and $t' = 1$ at the one-dimensional critical points A and B.

For $x = y = p$, the phase diagram in the $p-V_b$ plane is shown schematically in figure 2(b), and its features are entirely similar to those of the corresponding diagram for the undirected problem.

Since the H-shaped cell of figure 1(a) does not give rise to systematic errors for isotropic cases, we have used it, with adaptations to include bond directionality (Oliveira 1983), for the isotropic 'directed insulator' breakdown problem. For scaling factors $b = 2$ and 3 our results are, respectively, $p_c = 0.726$ and 0.663; $\nu = 1.398$ and 1.381; and $t' = 0.745$ and 0.821. Apart from the slight decrease in ν , these results seem to be consistent with those reported above.

In summary, we have proposed a small-cell PSRG to study the effects of concentration anisotropy and directionality in the dielectric breakdown problem on a two-dimensional square lattice. We have shown that, in spite of the anisotropies inherent in the problem, concentration anisotropy is irrelevant (in the RG sense), both for the undirected and directed cases, just as occurs in normal percolation. Relevant features of the phase diagrams in both the xy and $p-V_b$ planes have been discussed, and remarks have been made on the peculiarities of scaling theories when applied to breakdown problems. Directional effects have been included via 'directed insulators', an approximation which results in the appearance of a mixed (non-conducting) phase of conductors and 'mismatched' directed insulators.

We thank P M Oliveira and R Riera for useful remarks, and P M Duxbury for sending his results prior to publication. This work was supported in part by FINEP, CAPES and CNPq.

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